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1982 J. Phys. A: Math. Gen. 15 3685

(http://iopscience.iop.org/0305-4470/15/12/020)

J. Phys. A: Math. Gen. **15** (1982) 3685–3692. Printed in Great Britain

The Yang–Mills SU(2) equations of motion and conserved quantities on space-like infinity

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Received 28 May 1982

Abstract. Using the results of an approximation method for the Yang–Mills fields, we have calculated a conserved quantity corresponding to the charge at space-like infinity using a geometrical construction. The result obtained is what one would expect on physical grounds.

1. Introduction

Recently, Drechsler and Rosenblum (1981) proposed an approximation scheme to determine the classical equations of motion, including radiation reaction terms, for Yang–Mills SU(2) starting from the abelian Lienard–Wiechert-type solution for the gauge field produced by a moving point-like non-abelian charge $\bar{q}$. The higher-order iterations were expressed in terms of the regularised first-order Lienard–Wiechert solution. The dimensional regularisation procedure was introduced by Riesz and investigated in detail in Ma (1947), Schieve *et al* (1972) and Rosenblum (1981). It was applied to obtain approximate solutions of the nonlinear equations of motion for a charged Yang–Mills particle by starting from the known linear solution which is properly regularised to assume finite values on the world line of the particle. The Lorentz-type equations of motion were derived for the translational motion, including radiation reaction as well as the equations of motion for $\boldsymbol{q}$ in the internal isospace, the latter following from the covariant current conservation. As shown by Clarke and Rosenblum (1982), these types of equations have unique solutions for small angle scattering when suitable conditions are imposed in the infinite past.

In the abelian electromagnetic case, it is possible to define a conserved quantity, namely the charge at space-like infinity. Christodoulou (1982), using a geometrical construction, has given a definition of conserved quantities at space-like infinity. In this paper we explore the relationship between the approximation method with its resulting equations of motion and the exact definition of the conserved quantities. We first summarise the approximation method, and resulting equations of motion for Yang–Mills SU(2). We then give an introduction to the theory of conserved quantities at space-like infinity for Yang–Mills SU(2). Lastly, for the simple case of linearised Yang–Mills with two charges either parallel or antiparallel, the conserved quantities are computed explicitly.

2. Field equations and laws of motion

The field equations for the SU(2) gauge theory are given by

$$D^{\mu}\boldsymbol{F}_{\mu\nu}=\partial^{\mu}\boldsymbol{F}_{\mu\nu}-\boldsymbol{A}^{\mu}\times\boldsymbol{F}_{\mu\nu}=\boldsymbol{j}_{\nu} \tag{1}$$

with

$$\boldsymbol{F}_{\mu\nu}=\partial_{\mu}\boldsymbol{A}_{\nu}-\partial_{\nu}\boldsymbol{A}_{\mu}-\boldsymbol{A}_{\mu}\times\boldsymbol{A}_{\nu} \tag{2}$$

obeying the identities

$$D_{\{\kappa}\boldsymbol{F}_{\mu\nu\}}=D_{\kappa}\boldsymbol{F}_{\mu\nu}+D_{\mu}\boldsymbol{F}_{\nu\kappa}+D_{\nu}\boldsymbol{F}_{\kappa\mu}=0. \tag{3}$$

We use the metric $\eta_{\mu\nu}=\mathrm{diag}(1,-1,-1,-1)$ in Minkowski space. The coordinates are denoted by $x^{\mu}=(x^{0},x^{i})$, with $\partial_{\mu}=\partial/\partial x^{\mu}$. The argument x^{μ} of the gauge field strengths $\boldsymbol{F}_{\mu\nu}$, the potentials $\boldsymbol{A}_{\mu}$ or the current components $\boldsymbol{j}_{\nu}$ are frequently suppressed; $\times$ denotes the cross product in isospace, i.e. $(A^{\mu}\times F_{\mu\nu})_i=\varepsilon_{ijk}A_j^{\mu}(F_{\mu\nu})_k$ with ε_{ijk} being the Levi–Civita symbol.

For N classical point-like non-abelian charges the isovector source current is given by the expression

$$\boldsymbol{j}_{\nu}=\sum_{n=1}^{N}\int_{-\infty}^{+\infty}\boldsymbol{\tau}_{(n)\nu}\delta^{4}(x-z_{(n)}(\tau))\,\mathrm{d}\tau \tag{4}$$

where $z^{\mu}_{(n)}(\tau)$ denotes the trajectory of the nth particle as a function of the proper times τ, and $\boldsymbol{\tau}_{(n)}$ is an isovector space–time four-vector for the nth particle which can be shown to have the form

$$\boldsymbol{\tau}_{(n)}(\tau)=\boldsymbol{q}_{(n)}(\tau)u_{(n)}(\tau),\qquad u_{(n)}(\tau)=\mathrm{d}z_{(n)}(\tau)/\mathrm{d}\tau,$$

denoting the four-velocity of the nth particle at the proper time τ, with $\boldsymbol{q}_{(n)}(\tau)$ being the SU(2) charge at the proper time τ. As a consequence of the field equations (1), the current (4) is covariantly conserved, i.e.

$$D^{\nu}\boldsymbol{j}_{\nu}=\partial^{\nu}\boldsymbol{j}_{\nu}-\boldsymbol{A}^{\nu}\times\boldsymbol{j}_{\nu}=0. \tag{5}$$

Besides the source current $\boldsymbol{j}_{\nu}$ defined by (4) we introduce the symmetric energy–momentum tensor of matter distributed in the form of N non-abelian charged particles of masses m_n by

$$T^{\mu\nu}_{(\mathrm{M})}=\sum_{n=1}^{N}\int_{-\infty}^{+\infty}P^{\mu\nu}_{(n)}(\tau)\delta^{4}(x-z_{(n)}(\tau))\,\mathrm{d}\tau. \tag{6}$$

The explicit forms of the quantities $\boldsymbol{\tau}_{(n)}(\tau)$ and $P^{\mu\nu}_{(n)}(\tau)$ are determined as a consequence of energy and momentum conservation as well as covariant charge conservation.

In addition, we introduce

$$T^{\mu\nu}_{(\mathrm{F})}=\tfrac{1}{4}\eta^{\mu\nu}\boldsymbol{F}^{\kappa\lambda}\cdot\boldsymbol{F}_{\kappa\lambda}-\boldsymbol{F}^{\mu\kappa}\cdot\boldsymbol{F}^{\nu}{}_{\kappa} \tag{7}$$

where means inner product in charge space.

It is easy to show from equations (1) and (2) that the $T^{\mu\nu}_{(\mathrm{F})}$ obey

$$\partial_{\mu}T^{\mu\nu}_{(\mathrm{F})}=-\boldsymbol{F}^{\nu\mu}\cdot\boldsymbol{j}_{\mu}. \tag{8}$$

We require overall energy and momentum conservation

$$\partial_\mu (T^{\mu\nu}_{(\mathrm{M})} + T^{\mu\nu}_{(\mathrm{F})}) = 0. \tag{9}$$

Using equation (8), this can be rewritten in the form

$$\partial_\mu T^{\mu\nu}_{(\mathrm{M})} = \boldsymbol{F}^{\nu\mu} \cdot \boldsymbol{j}_\mu. \tag{10}$$

The equations (5) and (10) together with the expressions (4) and (6) are now the basis for the derivation of the laws of motion for the non-abelian point charges. To derive the respective equations we insert (4) and (6) into (5) and (10), respectively, multiply the first equation by an arbitrary smooth function $S(x)$ and the second by a set of arbitrary smooth functions $S_\nu(x)$, both of compact support, and integrate over all space–time. After an integration by parts, we break up $\bar{P}^{\mu\nu}_{(n)}$ and $\boldsymbol{\tau}_{(n)}$ into components parallel and perpendicular to the world line at the point determined by τ. With

$$n^\mu_{(n)} n_{(n)\mu} = -1, \qquad u^\mu_{(n)} n_{(n)\mu} = 0, \tag{11}$$

one has

$$\boldsymbol{\tau}^\nu_{(n)} = \boldsymbol{q}_{(n)} u^\nu_{(n)} + \boldsymbol{I}^\nu_{(n)}, \tag{12}$$

$$P^{\mu\nu}_{(n)} = m_{(n)} u^\mu_{(n)} u^\nu_{(n)} + a n^\mu_{(n)} u^\nu_{(n)} + a n^\nu_{(n)} u^\mu_{(n)} + \boldsymbol{R}^{\mu\nu}_{(n)}, \tag{13}$$

with

$$\boldsymbol{I}^\nu_{(n)} u_{(n)\nu} = 0, \qquad \boldsymbol{R}^{\mu\nu}_{(n)} u_{(n)\nu} = 0. \tag{14}$$

Using the expressions in equations (5) and (10) and integrating by parts, we obtain the final equation for each n:

$$\mathrm{d}\boldsymbol{q}_{(n)}/\mathrm{d}\tau = \boldsymbol{A}_\nu \times \boldsymbol{q}_{(n)} u^\nu_{(n)}, \tag{15}$$

$$\mathrm{d}(m_{(n)} u^\nu_{(n)})/\mathrm{d}\tau = \boldsymbol{q}_{(n)} \cdot \boldsymbol{F}^{\nu\mu} u_{(n)\mu}. \tag{16}$$

Equations (15) and (16) are exact, being consequences of the conservation laws for the non-abelian charge and for the total energy and momentum. However, the forms of the fields entering these equations are not known. They have to be determined from the nonlinear field equations (1). In the next section, the approximation method for the nonlinear field equations is presented.

3. The approximation method

We rewrite equation (1) as a second-order nonlinear equation for the potential in the form

$$\Box \boldsymbol{A}_\nu - \partial_\nu \partial^\mu \boldsymbol{A}_\mu = \boldsymbol{j}_\nu + \partial^\mu (\boldsymbol{A}_\mu \times \boldsymbol{A}_\nu) + \boldsymbol{A}^\mu \times \{\partial_\mu \boldsymbol{A}_\nu - \partial_\nu \boldsymbol{A}_\mu - \boldsymbol{A}_\mu \times \boldsymbol{A}_\nu\} \tag{17}$$

where $\Box = \partial^\mu \partial_\mu$ is the d'Alembert operator. We now expand $\boldsymbol{A}_\nu$ formally as

$$\boldsymbol{A}_\nu = \sum_m {}_{(m)}\boldsymbol{A}_\nu \tag{18}$$

where ${}_{(m)}\boldsymbol{A}_\nu$ is to be computed as a solution of a field equation containing nonlinear terms in the potentials up to the order $m-1$ as an effective current on the right-hand side. For example, the first- and second-order equations are

$$m=1: \qquad \Box_{(1)}\boldsymbol{A}_\nu - \partial_\nu \partial^\mu{}_{(1)}\boldsymbol{A}_\mu = \boldsymbol{j}_\nu, \tag{19}$$

$$m=2: \qquad \Box_{(2)}\boldsymbol{A}_\nu - \partial_\nu \partial^\mu_{(2)}\boldsymbol{A}_\mu = \partial^\mu_{(1)}\boldsymbol{A}_\mu \times {}_{(1)}\boldsymbol{A}_\nu + 2{}_{(1)}\boldsymbol{A}^\mu \times \partial_{\mu(1)}\boldsymbol{A}_\nu - {}_{(1)}\boldsymbol{A}^\mu \times \partial_{\nu(1)}\boldsymbol{A}_\mu. \tag{20}$$

Along with the potentials, we introduce the gauge fields

$$_{(m)}\boldsymbol{F}_{\mu\nu} = {}_{(m-1)}\boldsymbol{F}_{\mu\nu} + \partial_{\mu(m)}\boldsymbol{A}_{\nu} - \partial_{\nu(m)}\boldsymbol{A}_{\mu}.$$

If we require the Lorentz condition $\partial^{\mu}_{(1)}\boldsymbol{A}_{\mu} = 0$ we can choose for the solution of equation (19)

$$\boldsymbol{A}_{\mu} = (\boldsymbol{q}u_{\mu}/cp)_{\tau=\tau_r} \tag{21}$$

with $z^{\mu}(\tau_r)$ denoting the retarded point on the world line associated with the field point x^{μ} and with $p = (1/c)[x^{\mu} - z^{\mu}(\tau_r)]u_{\mu}(\tau_r)$. The Lorentz condition implies $\dot{\boldsymbol{q}} = 0$. To get a non-trivial result we must go to the next order and relax the lower-order restrictions but keeping the lower-order form of the potentials. For the rest of the paper we will assume that this is done.

4. The definition of charge

In the Yang–Mills theory it is possible to define a conserved quantity which is in some respects analogous to the electric charge in the Maxwell theory (Christodoulou 1982). In the Yang–Mills theory, however, in contrast to electromagnetism, this quantity may be different from zero even in the absence of sources. This is due to the fact that a Yang–Mills field is self interacting. Another difference between the abelian and non-abelian theories is that for the latter the conserved quantity can only be defined for a class of fields having special asymptotic behaviour at spatial infinity. To describe this asymptotic behaviour, we introduce in the exterior of the light cone of the origin in Minkowski space–time the space-like proper distance

$$\rho = (r^2 - t^2)^{1/2} \tag{22}$$

and the coordinates χ, ϑ, φ on the hyperboloid of unit space-like vectors where χ is defined by

$$t = \rho \sinh \chi, \qquad r = \rho \cosh \chi \tag{23}$$

and ϑ, φ are the usual angular coordinates on the two-sphere. The Minkowski metric is in these coordinates expressed as

$$\mathrm{d}s^2_{\mathrm{M}} = \mathrm{d}\rho^2 + \rho^2\, \mathrm{d}s^2_{\mathrm{H}} \tag{24}$$

where

$$\mathrm{d}s^2_{\mathrm{H}} = -\mathrm{d}\chi^2 + \cosh^2\chi\, (\mathrm{d}\vartheta^2 + \sin^2\vartheta\, \mathrm{d}\varphi^2) \tag{25}$$

is the metric on the hyperboloid $H \cong \mathbb{R}^1 \times S^2$. In general coordinates $\{s^a : a = 0, 1, 2\}$ on the hyperboloid we shall write

$$\mathrm{d}s^2_{\mathrm{H}} = \gamma_{ab}\, \mathrm{d}s^a\, \mathrm{d}s^b. \tag{26}$$

We consider then the class of Yang–Mills potentials whose components $\boldsymbol{A}_{\rho}$, $\boldsymbol{A}_{\chi}$, $\boldsymbol{A}_{\vartheta}$, $\boldsymbol{A}_{\varphi}$ in the above coordinate system have the following asymptotic properties:

$$\boldsymbol{A}_{\rho}(\rho, \chi, \vartheta, \varphi) = -\boldsymbol{\phi}(\chi, \vartheta, \varphi)\rho^{-1} + \mathrm{O}(\rho^{-1-\varepsilon}), \qquad \boldsymbol{A}_{\chi}(\rho, \chi, \vartheta, \varphi) = \boldsymbol{B}_{\chi}(\chi, \vartheta, \varphi) + \mathrm{O}(\rho^{-\varepsilon}),$$
$$\boldsymbol{A}_{\vartheta}(\rho, \chi, \vartheta, \varphi) = \boldsymbol{B}_{\vartheta}(\chi, \vartheta, \varphi) + \mathrm{O}(\rho^{-\varepsilon}), \qquad \boldsymbol{A}_{\varphi}(\rho, \chi, \vartheta, \varphi) = \boldsymbol{B}_{\varphi}(\chi, \vartheta, \varphi) + \mathrm{O}(\rho^{-\varepsilon}), \tag{27}$$

where $\varepsilon > 0$ and the symbol $O(\rho^{-\lambda})$ denotes a function of f with the property that there exists a constant C such that f satisfies

$$\rho^{\lambda}|f| \leqslant C,$$

and the partial derivatives of f with respect to the Cartesian coordinates $\{x^{\mu}\}$ satisfy

$$\rho^{\lambda+1}|\partial_{\mu} f| \leqslant C, \ldots, \rho^{\lambda+k}|\partial_{\mu_1} \ldots \partial_{\mu_k} f| \leqslant C \tag{28}$$

up to some finite order k.

The above asymptotic form of $\boldsymbol{A}$ is invariant under gauge transformations f which satisfy the asymptotic condition

$$f(\rho, \chi, \vartheta, \varphi) = g(\chi, \vartheta, \varphi) + O(\rho^{-\varepsilon}).$$

Under such gauge transformations the Lie algebra valued function $\boldsymbol{\phi}$ and the Lie algebra valued one-form $\boldsymbol{B} = \boldsymbol{B}_{\chi}\, \mathrm{d}\chi + \boldsymbol{B}_{\vartheta}\, \mathrm{d}\vartheta + \boldsymbol{B}_{\varphi}\, \mathrm{d}\varphi$ transform as

$$\boldsymbol{\phi} \mapsto g^{-1}\boldsymbol{\phi} g, \qquad \boldsymbol{B}_a \mapsto g^{-1}\boldsymbol{B}_a g + g^{-1}\partial_a g. \tag{29}$$

Thus $\boldsymbol{B}$ is a gauge potential and $\boldsymbol{\phi}$ a Higgs field on H. By virtue of the asymptotic hypothesis (27), the Yang–Mills equations on Minkowski space–time have a limit for $\rho \to \infty$ which is the following system of Yang–Mills–Higgs field equations on H:

$$\delta_{\gamma,\boldsymbol{B}} d_{\boldsymbol{B}} \boldsymbol{\phi} = 0, \qquad \delta_{\gamma,\boldsymbol{B}} \boldsymbol{G} = -\boldsymbol{\phi} \times d_{\boldsymbol{B}} \boldsymbol{\phi}, \tag{30}$$

where $\boldsymbol{G}$ is the field strength of the potential $\boldsymbol{B}$, $\boldsymbol{G}_{ab} = \partial_a \boldsymbol{B}_b - \partial_b \boldsymbol{B}_a - \boldsymbol{B}_a \times \boldsymbol{B}_b$, and $d_{\boldsymbol{B}}$ and $\delta_{\gamma,\boldsymbol{B}}$ denote the gauge covariant exterior derivative and the metric and gauge covariant divergence relative to the metric γ on H and the connection defined by the gauge potential $\boldsymbol{B}$.

By equations (30) the current I defined by

$$\boldsymbol{I} = d_{\boldsymbol{B}} \boldsymbol{\phi} \tag{31}$$

is covariantly conserved,

$$\delta_{\gamma,\boldsymbol{B}} \boldsymbol{I} = 0. \tag{32}$$

Suppose now that the connection defined by $\boldsymbol{B}$ has a continuous group of automorphisms Aut($\boldsymbol{B}$). Let $\boldsymbol{\psi}$ be a generator of automorphisms of $\boldsymbol{B}$, namely a Lie algebra valued function such that $d_{\boldsymbol{B}}\boldsymbol{\psi} = 0$. Then the current $\boldsymbol{\psi} \cdot \boldsymbol{I}$ (where $\cdot$ denotes the bivariant inner product in the Lie algebra) is covariantly conserved and, being a scalar relative to the gauge group, is conserved in the ordinary sense: $\delta_{\gamma}(\boldsymbol{\psi} \cdot \boldsymbol{I}) = 0$. Therefore the integral of this scalar current on any section $\Sigma \cong S^2$ of the hyperboloid $H \cong \mathbb{R}^1 \times S^2$ is the same. P is therefore a conserved quantity,

$$P(\boldsymbol{\psi}) = \int_{\Sigma} \boldsymbol{\psi} \cdot \boldsymbol{I}^a \, \mathrm{d}S_a. \tag{33}$$

Now let us assume that the connection defined by $\boldsymbol{B}$ is flat, that is $\boldsymbol{G} = 0$. Then there is an isomorphism between Aut($\boldsymbol{B}$) and the gauge group. Hence there is a linear map σ between the Lie algebra of the gauge group and the space of generators of automorphisms of $\boldsymbol{B}$. Therefore we can define through (33) a linear function, that is, a one-form, Q on the Lie algebra, $Q(\boldsymbol{e}) = P(\sigma(\boldsymbol{e}))$ which is conserved. This is Q, the Yang–Mills charge.

We have shown that we can define the Yang–Mills charge Q as q one-forms on the Lie algebra, if the field strength $\boldsymbol{G}$ of the gauge potential $\boldsymbol{B}$ on the hyperboloid H vanishes. By equations (30), $\boldsymbol{G}=0$ implies

$$\boldsymbol{\phi}\times d_{\boldsymbol{B}}\boldsymbol{\phi}=0. \tag{34}$$

$\boldsymbol{G}=0$ implies also that we can find a gauge in which $\boldsymbol{B}=0$. Equation (34) implies that in this gauge $\boldsymbol{\phi}$ takes values in a one-dimensional subalgebra of the Lie algebra of SU(2). Thus there is some element $\boldsymbol{e}_0$ in the Lie algebra of SU(2) such that

$$\boldsymbol{\phi}=f\boldsymbol{e}_0$$

where f is an ordinary function on the hyperboloid H. Then equations (30) reduce to

$$\Box_\gamma f=0$$

where

$$\Box_\gamma=\delta_\gamma d$$

is the d'Alembert operator of the metric γ on H.

5. Calculation of the total charge

We now consider two point charges $\boldsymbol{q}_1$, $\boldsymbol{q}_2$ coming in from infinity with impact parameter a. The zeroth-order motion in the centre of mass is

$$\begin{aligned}
&z^1_{(1)}=u\tau/(1-u^2)^{1/2}, \qquad z^1_{(2)}=-u\tau/(1-u^2)^{1/2},\\
&z^2_{(1)}=-a/2, \qquad z^2_{(2)}=a/2, \qquad z^3_{(1)}=0, \qquad z^3_{(2)}=0,\\
&z^0_{(1)}=\tau/(1-u^2)^{1/2}, \qquad z^0_{(2)}=\tau/(1-u^2)^{1/2}
\end{aligned} \tag{35}$$

The retarded proper times τ_1 and τ_2 are determined by the equations

$$\begin{aligned}
&\eta_{\mu\nu}(x^\mu-z^\mu_{(1)}(\tau_1))(x^\nu-z^\nu_{(1)}(\tau_1))=0,\\
&\eta_{\mu\nu}(x^\mu-z^\mu_{(2)}(\tau_2))(x^\nu-z^\nu_{(2)}(\tau_2))=0,\\
&(x^0-z^0_1)>0, \qquad (x^0-z^0_2)>0.
\end{aligned} \tag{36}$$

We have

$$\tau_1=\frac{t-x^1u}{(1-u^2)^{1/2}}-\left[\left(\frac{t-x^1u}{(1-u^2)^{1/2}}\right)^2+(x^1)^2+(x^2+\tfrac{1}{2}a)^2+(x^3)^2-t^2\right]^{1/2},$$

$$\tau_2=\frac{t+x^1u}{(1-u^2)^{1/2}}-\left[\left(\frac{t+x^1u}{(1-u^2)^{1/2}}\right)^2+(x^1)^2+(x^2-\tfrac{1}{2}a)^2+(x^3)^2-t^2\right]^{1/2}$$

and

$$p_1=\tau_1-(t-ux^1)/(1-u^2)^{1/2}, \qquad p_2=\tau_2-(t+ux^1)/(1-u^2)^{1/2}. \tag{37}$$

The zeroth-order potential is then given by

$$\boldsymbol{A}_0=\frac{-\boldsymbol{q}_1}{p_1(1-u^2)^{1/2}}-\frac{\boldsymbol{q}_2}{p_2(1-u^2)^{1/2}},$$

$$\boldsymbol{A}_1=\frac{\boldsymbol{q}_1u}{p_1(1-u^2)^{1/2}}-\frac{\boldsymbol{q}_2u}{p_2(1-u^2)^{1/2}}, \qquad \boldsymbol{A}_2=\boldsymbol{A}_3=0, \tag{38}$$

where $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$ are constants.

We find that this potential satisfies the asymptotic conditions (27) and the fields $\boldsymbol{\phi}$ and $\boldsymbol{B}$ on the hyperboloid H are given by

$$\boldsymbol{\phi}=\frac{\alpha}{(\alpha^2+1-u^2)^{1/2}}\boldsymbol{q}_1+\frac{\beta}{(\beta^2+1-u^2)^{1/2}}\boldsymbol{q}_2, \tag{39}$$

$$\boldsymbol{B}=\frac{\boldsymbol{q}_1\,\mathrm{d}\alpha}{(\alpha^2+1-u^2)^{1/2}}+\frac{\boldsymbol{q}_2\,\mathrm{d}\beta}{(\beta^2+1-u^2)^{1/2}}, \tag{40}$$

where

$$\begin{aligned}&\alpha=\sinh\chi-u\cos\vartheta\cosh\chi, \qquad \beta=\sinh\chi+u\cos\vartheta\cosh\chi\\ &(x^1=r\cos\vartheta, \qquad x^2=r\sin\vartheta\cos\varphi, \qquad x^3=r\sin\vartheta\sin\varphi.\end{aligned} \tag{41}$$

As $\mathrm{d}\boldsymbol{B}=0$, the field strength of $\boldsymbol{B}$ is

$$\boldsymbol{G}=-\frac{\boldsymbol{q}_1\times\boldsymbol{q}_2\,\mathrm{d}\alpha\wedge\mathrm{d}\beta}{(\alpha^2+1-u^2)^{1/2}(\beta^2+1-u^2)^{1/2}}. \tag{42}$$

The current $\boldsymbol{I}$ is given by

$$\begin{aligned}\boldsymbol{I}=d_{\boldsymbol{B}}\boldsymbol{\phi}&=\mathrm{d}\boldsymbol{\phi}-\boldsymbol{B}\times\boldsymbol{\phi}\\ &=\frac{(1-u^2)}{(\alpha^2+1-u^2)^{3/2}}\boldsymbol{q}_1\,\mathrm{d}\alpha+\frac{(1-u^2)}{(\beta^2+1-u^2)^{3/2}}\boldsymbol{q}_2\,\mathrm{d}\beta\\ &\quad-\boldsymbol{q}_1\times\boldsymbol{q}_2\left(\frac{\beta}{(\beta^2+1-u^2)^{1/2}}\frac{\mathrm{d}\alpha}{(\alpha^2+1-u^2)^{1/2}}\right.\\ &\quad\left.-\frac{\alpha}{(\alpha^2+1-u^2)^{1/2}}\frac{\mathrm{d}\beta}{(\beta^2+1-u^2)^{1/2}}\right).\end{aligned} \tag{43}$$

If the charges $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$ are parallel,

$$\boldsymbol{q}_1=k_1\boldsymbol{e}_3, \qquad \boldsymbol{q}_2=k_2\boldsymbol{e}_3, \tag{44}$$

where $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ is an orthonormal basis in the Lie algebra, we have $\boldsymbol{q}_1\times\boldsymbol{q}_2=0$ and expressions (39), (40), (42), (43) reduce to

$$\boldsymbol{\phi}=\boldsymbol{e}_3 g, \tag{45}$$

$$\boldsymbol{B}=\boldsymbol{e}_3\,\mathrm{d}f, \tag{46}$$

$$\boldsymbol{G}=0, \tag{47}$$

$$\boldsymbol{I}=\boldsymbol{e}_3\,\mathrm{d}g, \tag{48}$$

where

$$f=k_1\sinh^{-1}(\alpha/(1-u^2)^{1/2})+k_2\sinh^{-1}(\beta/(1-u^2)^{1/2}), \tag{49}$$

$$g=\frac{k_1\alpha}{(\alpha^2+1-u^2)^{1/2}}+\frac{k_2\beta}{(\beta^2+1-u^2)^{1/2}}. \tag{50}$$

Equations (30) are then found to be satisfied and we have a flat connection $\boldsymbol{B}$. There are three linearly independent solutions ${}^{(1)}\boldsymbol{\psi}$, ${}^{(2)}\boldsymbol{\psi}$, ${}^{(3)}\boldsymbol{\psi}$ of the generator of automorphisms equation:

$$d_{\boldsymbol{B}}\boldsymbol{\psi} = \mathrm{d}\boldsymbol{\psi} - \boldsymbol{B} \times \boldsymbol{\psi} = 0, \qquad {}^{(1)}\boldsymbol{\psi} = \boldsymbol{e}_3,$$
$${}^{(2)}\boldsymbol{\psi} = \sin f\,\boldsymbol{e}_1 + \cos f\,\boldsymbol{e}_2, \qquad {}^{(3)}\boldsymbol{\psi} = \cos f\,\boldsymbol{e}_1 - \sin f\,\boldsymbol{e}_2. \tag{51}$$

The conserved charge $\boldsymbol{Q}$ is then

$$\boldsymbol{Q} = Q^i\boldsymbol{e}_i \tag{52}$$

where

$$Q^i = \int_\Sigma {}^{(i)}\boldsymbol{\psi} \cdot \boldsymbol{I}^a \,\mathrm{d}S_a$$

that is

$$Q^1 = Q^2 = 0, \qquad Q^3 = \int_\Sigma \partial^a g \,\mathrm{d}S_a. \tag{53}$$

Calculating the integral Q^3 for the section $\chi = 0$, we obtain

$$Q^3 = 4\pi(k_1 + k_2) \tag{54}$$

and thus

$$\boldsymbol{Q} = 4\pi(\boldsymbol{q}_1 + \boldsymbol{q}_2). \tag{55}$$

6. Conclusions

Using the results of an approximation method for the Yang–Mills fields we have calculated the conserved quantity at space-like infinity using a geometrical construction. The result obtained is what one would expect on physical grounds.

Acknowledgment

We would like to thank Professor Peter Havas for useful critical remarks. One of us, Arnold Rosenblum, would like to thank the National Science Foundation under grant PHY 8008313 for support.

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